» Tangents & Normals

The **tangent to a curve** is the straight line that 'just touches' the curve at a given point (from Latin *TANGERE*, meaning 'to touch' – from which we get words like *tangible*). A more technical definition is that a tangent's gradient is equal to the curve's derivative at its point of intersection.

Therefore, the existence of a tangent is usually contingent on the existence of a derivative at the given point, since the tangent and curve (by definition) share the same gradient at that point. For instance, $y = |x|$ has no derivative at $x = 0$ and thus has no tangent at that point.

The exception is when the tangent to a curve is *vertical* at a given point. There is no gradient defined at such points (e.g. $y = \sqrt[3]{x}$ at $x = 0$), but the tangent does exist (it is simply a vertical line).

I think, therefore I want to find tangents

The invention of calculus rendered the calculation of derivatives (and thus gradients) a relatively easy process. But before calculus was a well-established branch of maths, the locating of tangents was a notoriously difficult conundrum in mathematics. People could make guesses at the equation of a tangent, but could never be completely accurate in their estimates. Rene Descartes, who famously said "I think, therefore I am", said about the calculation of tangents:

> *And I dare say that this is not only the **most useful** and **most general** problem in geometry that I know, but even that I have **ever desired to know**.*

» How to calculate a tangent

'Calculating a tangent' (or a normal) is really short-hand for 'calculating the equation of a tangent'. Remembering that tangents are essentially straight lines, recognise the fact that what we are aiming to end up with is something in the form $y = mx + b$. A typical question might be phrased like this:

Calculate the tangent to $y = \sin 3x$ at $x = {}^{\pi}/_{3}$.

The question is so bare that you might not know where to start. That's okay – you'll soon see that solving this question can be broken down into three easy steps.

The first thing we should consider is what result we are trying to get to, because that will govern the means we choose to get there. We want to find the equation of a **straight line** through a **particular point**, with a **certain gradient** (equal to the gradient of the curve at that point). Putting this all together means that we will be using the **point-gradient formula** for a straight line, $y - y_1 = m(x - x_1)$.

Notice that the question has already given us the value of x_1 (in this case, ${}^{\pi}/_{3}$). Our approach to this question will therefore lead us to work out the other unknowns in this equation – namely y_1 and m.

» Step 1: Calculate y_1

Let's start off by working out the value of y_1. This is simply the y-value that corresponds to x_1. So substitute the given value for x_1 (${}^{\pi}/_{3}$) into the equation of the curve:

$$\begin{aligned} y_1 &= \sin 3x_1 \\ &= \sin \frac{3\pi}{3} \\ &= \sin \pi \\ y_1 &= 0 \end{aligned}$$

That wasn't so hard! In your working out, underline or highlight this result in some way, so that you can know where it is when we come back to it later.

» Step 2: Calculate m

This is where the rubber hits the road with all the rules of differentiation you have been learning. By now we should be able to do this step in our sleep!

$$y = \sin f(x)$$
$$y' = f'(x)\cos f(x)$$
$$= 3\cos 3x$$

Now, notice that y' is in terms of x. This means we can calculate the value of m anywhere along our curve. But we don't want it at just anywhere – we want to calculate it at x_1. So, in the same way that we found y_1 by substituting x_1 into y, we find m by substituting x_1 into y'.

$$m = 3\cos 3x_1$$
$$= 3\cos\frac{3\pi}{3}$$
$$= 3\cos\pi$$
$$m = -3$$

Highlight this result too. Now we have all the pieces – we just need to combine them, which is the last step.

» Step 3: Point-Gradient formula

All that's required now is some simple substitution into the point-gradient formula that we mentioned earlier.

$$y - y_1 = m(x - x_1)$$
$$y - 0 = -3(x - {}^{\pi}/_{3})$$
$$y = -3x + \pi$$

Et voila. There's the equation of our tangent. If the question asked for us to calculate the equation of the **normal** instead of the tangent to the curve, our method would only needed to be adjusted slightly: when calculating m, we take the **negative reciprocal** of the gradient that is calculated, resulting in the straight line being perpendicular to the curve rather than parallel. Every other step is identical.

» Exercises

1. $y = e^{2x}$ at $x = 1$
2. $y = 2x^{(-2)}$ at $x = \sqrt[3]{4}$
3. $y = \ln\left(\frac{x}{2}\right)$ at $x = 10$
4. $y = \tan\left(\frac{x}{3}\right)$ at $x = 135°$
5. $y = \sin^2 x$ at $x = \pi^c$
6. $y = \sec 2x$ at $x = \frac{\pi^c}{6}$